MATH4210: Financial Mathematics Tutorial 5

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Convergence of r.v.s

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. X and $\{X_n\}$ are $\mathbb{R}$ valued (sequence of) r.v.s.

almost everywhere

Definition (Convergence almost surely)

Denote by $X_n \to X$ a.s. (almost surely) if

$$\mathbb{P}[\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}] = 1$$

measure

Definition (Convergence in Probability)

Denote by $X_n \to X$ in probability if for any $\rho > 0$

$$\lim_{n\to\infty} \mathbb{P}[\{\omega \in \Omega : |X_n(\omega) - X(\omega)| \geq \rho\}] = 0$$

Convergence of r.v.s

Proposition

$X_n \to X$ a.s. implies $X_n \to X$ in probability.

Proposition (admit)

$X_n \to X$ in probability implies there exists a subsequence of X_n converging to X a.s..

Definition (Convergence in Law (in Distribution))

Let F_n and F be the c.d.f. of X_n and X for all $n \in \mathbb{N}$. $X_n \to X$ in Law (in Distribution) if

$$\lim_{n\to\infty} F_n(x) = F(x)$$

for any $x \in \mathbb{R}$ where F is continuous at x.

Proposition (admit)

$X_n \to X$ in probability implies $X_n \to X$ in Law.

$\mathbb{L}^p$ - convergence.

Definition

Given $p > 0$, denote by $X_n \to X$ in L^p if

$$\lim_{n\to\infty} \mathbb{E}[|X_n - X|^p] = 0$$

Question

(a). Show that $X_n \to X$ in L^2 implies $X_n \to X$ in L^1.
(b). Show that $X_n \to X$ in L^p implies $X_n \to X$ in probability.

(a) $X_n \xrightarrow{\mathbb{L}^2} X$ if $\lim_{n\to+\infty} \mathbb{E}[\,|X_n - X|^2\,] = 0$

$X_n \xrightarrow{\mathbb{L}^1} X$ if $\lim_{n\to+\infty} \mathbb{E}[\,|X_n - X|\,] = 0$

CV a.s. $\Rightarrow$ CV in probability.

Conditions

subsequence

Monotone CV theorem

Dominated CV theorem

⋮

$\Downarrow$

CV in law.

CV in $\mathbb{L}^p$

Assume $X_n \xrightarrow{L^2} X$. So by definition:

$$\lim_{n\to+\infty} \mathbb{E}[|X_n - X|^2] = 0$$

Fix $n \in \mathbb{N}$.

$\mathbb{E}[|X_n - X|] = \mathbb{E}[\sqrt{|X_n - X|^2}]$ ... $(*)$.

Jensen's inequality:

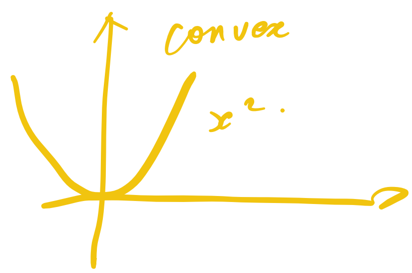
For f convex, then $\mathbb{E}[f(X)] \geq f(\mathbb{E}[X])$.

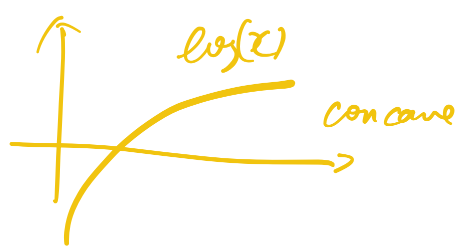
and for f concave, then $\mathbb{E}[f(X)] \leq f(\mathbb{E}[X])$

Recall the variance of r.v.

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 \geq 0$$

$$\Rightarrow \mathbb{E}[X^2] \geq \mathbb{E}[X]^2.$$





So $(*) = \mathbb{E}[\sqrt{|X_n - X|^2}]$

$$\leq \sqrt{\mathbb{E}[|X_n - X|^2]}$$

Since $x \mapsto \sqrt{x}$ is continuous.

So $\lim_{n\to+\infty} \sqrt{\mathbb{E}[|X_n - X|^2]} = \sqrt{\lim_{n\to+\infty} \mathbb{E}[|X_n - X|^2]}$

$$= 0$$

$\square$

(b) Fix $\rho > 0$. $n \in \mathbb{N}$.

Markov's inequality:

$\forall k > 0$. $\mathbb{P}[\{|Y| > k\}] \leq \frac{1}{k} \cdot \mathbb{E}[|Y|]$.

$\mathbb{P}[\{\omega \in \Omega : |X_n(\omega) - X(\omega)| > \rho\}]$

$= \mathbb{P}[\{|X_n - X| > \rho\}]$ (simplification)

$= \mathbb{P}[\{\underbrace{|X_n - X|^p}_{Y} \geq \underbrace{\rho^p}_{k}\}]$ in Markov's inequality

$\leq \frac{1}{\rho^p} \cdot \mathbb{E}[|X_n - X|^p]$

$\xrightarrow{n\to+\infty} 0$ since $X_n \xrightarrow{L^p} X$ $\square$.

Brownian Motions

Question

Let $f : \mathbb{R} \to \mathbb{R}$ be such that for some constant $C > 0$, $f(x) < e^{C|x|}$ for all $x \in \mathbb{R}$. Define

$$u(t,x) = \underbrace{\mathbb{E}[f(B_T)|B_t = x]}_{\text{random}} = \underbrace{\mathbb{E}[f(B_T - B_t + x)]}_{\text{deterministic.}}.$$

Show that

(a)

$$\partial_x u(t,x) = \mathbb{E}[\frac{B_T - B_t}{T - t} f(B_T - B_t + x)]$$

(b)

$$\partial_x^2 u(t,x) = \mathbb{E}[\frac{(B_T - B_t)^2 + (T - t)}{(T - t)^2} f(B_T - B_t + x)]$$

(a). $u(t,x) = \mathbb{E}[f(B_T - B_t + x)]$.

Recall that $B_T - B_t \sim N(0, T-t)$. So let $Y \sim N(0, T-t)$.

Then $u(t,x) = \mathbb{E}[f(Y+x)]$

$$= \int_{\mathbb{R}} f(y+x) \cdot p_Y(y)\, dy.$$

where p_Y is the pdf of Y: $\forall y \in \mathbb{R}$, $p_Y(y) = \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{y^2}{2(T-t)}}$

Then: $\partial_x u(t,x) = \int_{\mathbb{R}} f'(y+x) \cdot p_Y(y)\, dy.$

$$= \underbrace{\Big[f(y+x) \cdot p_Y(y) \Big]_{y \to -\infty}^{y \to +\infty}}_{0} - \int_{\mathbb{R}} f(y+x) p_Y'(y)\, dy.$$

Notice $p_Y'(y) = \frac{1}{\sqrt{2\pi(T-t)}} \cdot \left(-\frac{2y}{2(T-t)}\right) \cdot e^{-\frac{y^2}{2(T-t)}}$

$$= -\frac{y}{T-t} \cdot p_Y(y)$$

Therefore: $\partial_x u(t,x) = + \int_{\mathbb{R}} \cdot \underbrace{+\frac{y}{T-t} f(y+x)}_{F(y)} p_Y(y)\, dy.$ $= \int_{\mathbb{R}} F(y) p_Y(y)\, dy$

$= \mathbb{E}[F(Y)]$

$F(y) = \frac{y}{T-t} f(y+x)$

$$= \mathbb{E}\left[\frac{Y}{T-t} f(Y+x)\right]$$

$$= \mathbb{E}\left[\frac{B_T - B_t}{T-t} f(B_T - B_t + x)\right]$$

(b) similarly, apply integration by part . . . ~

N is the cdf of $N(0,1)$ $\quad N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy.$

d_1 (or d_2) are functions of stock price.

$$d_1(x) = \frac{\log(x/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

Question

Consider the European call option price at time t:

$$C_E(t,x) = xN(d_1(x)) - e^{-r(T-t)}KN(d_2(x))$$

$$C_E(t,S_t) = S_tN(d_1) - e^{-r(T-t)}KN(d_2)$$

where $d_1 = \frac{ln(S_t/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$ *and* $d_2 = d_1 - \sigma\sqrt{T-t}$*. Compute*

(a) Delta: $\Delta = \partial_x C_E(t,S_t)$*.* Rho, Theta.

(b) Gamma: $\Gamma = \partial_x^2 C_E(t,S_t)$*.*

denote $\tau = T-t$

(a). $\partial_x d_1(S_t) = \frac{1}{\sigma\sqrt{\tau}} \cdot \frac{1}{\cancel{K}} \cdot \frac{\cancel{K}}{S_t} = \frac{1}{S_t\sigma\sqrt{\tau}} = \partial_x d_2(x).$

And $\partial_x N(d_1(S_t)) = \partial_x d_1(S_t) \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2(S_t)}{2}}$

So $\Delta = \partial_x C_E(t, S_t)$

$$= N(d_1(S_t)) + S_t \cdot \partial_x N(d_1(S_t)) - e^{-r\tau} K \cdot \partial_x N(d_2(S_t))$$

$$= N(d_1(S_t)) + S_t \cdot \partial_x d_1(S_t) \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2(S_t)}{2}} - e^{-r\tau} K \cdot \partial_x d_1(S_t) \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2(S_t)}{2}}$$

$$= N(d_1(S_t)) + \frac{1}{\sqrt{2\pi}} \partial_x d_1(S_t) \left(S_t e^{-\frac{d_1^2(S_t)}{2}} - K e^{-r\tau} e^{-\frac{d_2^2(S_t)}{2}} \right)$$

$$= \underline{N(d_1(S_t))} + \frac{1}{\sqrt{2\pi}} \partial_x d_1(S_t) \cdot e^{-\frac{d_1^2(S_t)}{2}} \Big(\underbrace{S_t - K e^{-r\tau} e^{-\frac{d_1^2(S_t) - d_2^2(S_t)}{2}}}_{\text{"}0\text{"}} \Big)$$

implies

Notice: $d_1^2(S_t) - d_2^2(S_t)$

$$= (d_1 + d_2)(d_1 - d_2)$$

$$= \cancel{\sigma\sqrt{\tau}} \cdot \left(\frac{2\log(S_t/K) + 2r\tau}{\cancel{\sigma\sqrt{\tau}}} \right)$$

$$= \underline{2\log(S_t/K) + 2r\tau}.$$

Thus: $S_t - K e^{-r\tau} \cdot e^{-\frac{d_1^2 - d_2^2}{2}} = S_t - K \cancel{e^{-r\tau}} \cdot e^{\log S_t/K + \cancel{r\tau}}$

$$= S_t - K \cdot S_t/K$$

$$= 0$$

$\Rightarrow \Delta = N(d_1(S_t))$

(b): $\Gamma = \partial_{xx}^2 C_E(t, S_t) = \text{"}\partial_x \Delta\text{"}$

$$= \partial_x N(d_1(S_t))$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \cdot \partial_x d_1(S_t)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \cdot \frac{1}{\sigma\sqrt{\tau} \cdot S_t}$$

$$= p_Z(d_1) \cdot \frac{1}{\sigma\sqrt{\tau}\, S_t} \quad \text{where } p_Z \text{ is pdf of } N(0,1)$$